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Approximation of unbounded functions via compactification

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Abstract

Approximants to functions $f(s)$ that are allowed to possess infinite limits on their interval of definition, are constructed.

To this end a compactification of $\mathbb{R}^n$ is developed which is based on the projection of $\mathbb{R}^n$ on a bowl-shaped subset of a parabolic surface. This compactification induces a bijection and a metric with several desirable properties that make it a useful tool for rational approximation of unbounded functions.

Roughly speaking this compactification enables us to show that unbounded functions can be approximated by rational functions on a closed interval; thus we also obtain an extension to Weierstrass' celebrated theorem. An extension to a Fourier-type theorem is also obtained. Roughly speaking, our result states that unbounded periodic functions can be approximated by quotients of certain trigonometric sums. The characteristics of the main results are the following. The approximations do not require the original approximated function to possess a restricted rate of growth. Neither do they require that the approximated function possess any amount of smoothness. Moreover, the numerator and denominator, in an approximating quotient are guaranteed not to vanish simultaneously. Furthermore, some of the proposed approximations are guaranteed to be bounded at every point at which the original approximated function is bounded. Beside the tool of compactification we also employ Bernstein polynomials and Cesaro means of "trigonometric sums".

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1. Introduction

The purpose of this study is to offer approximations to unbounded functions and to unbounded vector functions. Some of the celebrated results in approximation theory include Weierstrass' and Fourier-type theorems. Weierstrass' theorem states that every function $f(s) \in C[a,b]$ can be uniformly approximated by polynomials. A Fourier-type theorem (roughly speaking) states that a continuous periodic function on $(-\infty, \infty)$ can be approximated by a quotient of "trigonometric sums". The features of our approximations are as follows.

The approximations do not impose restrictions on the order of growth of the unbounded approximated function at its singular points. Neither do they require analyticity or high order of smoothness from the approximated functions $f(s)$. Moreover, the numerator and denominator, in an approximating quotient are guaranteed not to vanish simultaneously. Furthermore, the proposed approximations are guaranteed to be bounded at every point where the original approximated function is bounded. Usually, the conventional approximations of unbounded functions imposes restrictions on the order of growth of the approximated functions at their singular points. See e.g. [1,3–5 ,8,9,11,12]. Usually, an approximated function needs to possess a certain amount of smoothness. This is the case in the theory of Pade approximants. Functions must be meromorphic or "somehow related" to a meromorphic function. This is also the case in the theory of Jacobi and other orthogonal polynomials. For example, in the case of Jacobi polynomials the given approximated function $f(s)$ on $[-1, 1]$ cannot grow faster, at $s = \pm 1$, than $(1-s)^{\alpha}(1+s)^{\beta}$ for some α, β real. It is noteworthy that there is no guarantee that the numerator and denominator, in a Pade approximant will not vanish simultaneously. Neither is there guarantee that a Pade approximant will be bounded at all points where the approximated function is bounded.

The following definition describes the nature of functions that are called "continuouslike accepting infinitudes". These are the generic functions to be approximated in the current study.

Definition 1. The function f is called continuouslike accepting infinitudes on $[a,b]$, in short, $f(s) \in CAI[a,b]$, if for every $\widehat{s} \in [a,b]$ one of the two conditions are met;
(a) f is continuous at $\widehat{s} \in [a,b]$ or
(b) f is discontinuous at $\hat{s} \in [a,b]$, but there exists $\lim_{s\to\widehat{s}} f(s)$, and then such $\widehat{s}$ satisfies that

$$\lim_{s\to\hat{s}} f(s) = \infty \text{ or } \lim_{s\to\hat{s}} f(s) = -\infty. \tag{1.1}$$

If $\widehat{s} = a$ (respectively, $\widehat{s} = b$) we assume that

$$f(a^+) = \infty \text{ or } f(a^+) = -\infty, \text{ respectively,} \\ f(b^-) = \infty \text{ or } f(b^-) = -\infty. \tag{1.2}$$

Notice that f continuous at $\hat{s} = a$ (respectively, f continuous at $\hat{s} = b$) means as usual, there exists $f(a^+) := \lim_{s\to a^+} f(s)$ and $f(a) = f(a^+)$ (respectively, there exists $f(b^-) := \lim_{s\to b^-} f(s)$ and $f(b) = f(b^-)$).

The vector function $V(s) = \langle v_1(s), v_2(s), \ldots, v_m(s)\rangle$, is said to be continuouslike accepting infinitudes on $[a, b]$, in short

$$V(s) \in CAI[a, b], \text{ if each component } v_k(s) \in CAI[a, b], k = 1, 2, \ldots, m.$$

It is evident from the definition that we impose no restriction on the rate of growth of a function $f(s) \in CAI[a, b]$, at its singular points.

The main tool for obtaining the extensions to Weierstrass' and Fourier's theorem is the parabolic compactification, to be developed in the sequel.

We denote a sequence of generic approximants by $A(f, n, s)$ or by $\hat{A}(f, n, s)$, $n = 1, 2, 3, \ldots$.

The order of events in this work is as follows. In Section 1 we construct certain mappings from a certain set, to be called the "ultra extended $\mathbb{R}^n$", to a parabolic surface $x_{n+1} = x_1^2 + x_2^2 + \cdots + x_n^2$. Two bijections from the "ultra extended $\mathbb{R}^n$" are constructed. One of the bijections maps the "ultra extended $\mathbb{R}^n$" to a bowl-shaped set on the parabolic surface. In Section 3, we construct a metric that is induced by the compactification in Section 2. In Section 4 we formulate and prove a theorem about generic approximations to functions that are continuouslike accepting infinitudes on an interval $[a, b]$. Extensions to Weierstrass' and Fourier's theorem are also obtained. In Section 5, we discuss approximants $A(f, n, s)$ that are "N totally compatible" with a function $f(s) \in CAI[a, b]$. These approximants $A(f, n, s)$, $n = N$, $N + 1$,... are such that $|A(f, n, s)| \neq \infty$ whenever $|f(s)| \neq \infty$. They could be useful in theoretical and practical considerations when delegating to a digital computer the task of numerical approximations of unbounded functions. We also consider the approximation of periodic functions $f(s)$ on $(-\infty, \infty)$ that belong to $CAI[-\pi, \pi]$ and $f(s + 2\pi) = f(s)$. Ultimately, we consider vector functions $V(s) = \langle v_1(s), \ldots, v_m(s)\rangle \in CAI[a, b]$.

The development of the compactification here has been influenced by the work [6]. In [6] the complex plane is supplemented by a continuum of ideal points that account for "all directions at infinity". The complex plane is then mapped onto a spherical bowl. Its boundary is the image of the continuum of ideal points supplementing the complex plane. Topologically the compactification employed in this work is equivalent to the compactification developed in [6]. Unfortunately, the compactification in [6], possess radicals that prevent it of being a tool for rational approximations.

2. Bijections induced by the parabolic compactification

Consider the set of points in $\mathbb{R}^n$ given by

$$\mathbb{R}^n = \{z | z = (\alpha_1, \ldots, \alpha_n), -\infty < \alpha_j < \infty, j = 1, 2, \ldots n\}, \tag{2.1}$$

together with the ideal set *ID* defined by

$$ID := \{\infty u | u \in \mathbb{R}^n \text{ and } 1 = ||u||^2 = u_1^2 + u_2^2 + \cdots + u_n^2\}. \tag{2.2}$$

The set $\mathbb{R}^n \cup ID$ will be called the ultra extended $\mathbb{R}^n$.

Consider a fixed point $P \in \mathbb{R}^{n+1}$, $P = (0, 0, \ldots, 0, \gamma)$, with $\gamma > 0$ a fixed real number. Henceforth we identify a point $z = (\alpha_1, \ldots, \alpha_n)$ with a point

$Q \in \mathbb{R}^{n+1}$ where $Q = (\alpha_1, \ldots, \alpha_n, 0)$. The coordinates of a point $Z = (x_1, \ldots, x_n, x_{n+1})$ on the straight line PQ is given for some real number t by the formula

$$\overrightarrow{PZ} = t\overrightarrow{PQ}. \tag{2.3}$$

The vector relation (2.3) is equivalent to

$$x_1 = t\alpha_1, \ldots, x_n = t\alpha_n, \quad x_{n+1} - \gamma = -t\gamma. \tag{2.4}$$

The straight line *PQ* intersects the parabolic surface

$$x_{n+1} = x_1^2 + \cdots + x_n^2 \tag{2.5}$$

at two points. One of the two points, say Z, where the line *PQ* intercepts the parabolic surface is "between" *P* and *Q*. Then $0 \leqslant t \leqslant 1$ in the relation (2.3). If *Z* "is not between" *P* and *Q* then t in (2.3) is negative. In this manner two images of the ultra extended $\mathbb{R}^n$ are generated on the parabolic surface. One of the images is a bounded parabolic bowl that is a closed set. The other image, that is unbounded, shares a circle as a common boundary with the parabolic bowl. This circle is the image of the ideal points augmenting the set of $\mathbb{R}^n$ with all "directions at infinity". We can thus determine two bijections between the points $Q \in \mathbb{R}^n \cup ID$ and certain subsets of the parabolic bowl as described above.

From (2.4) we obtain after squaring that

$$x_1^2 = t^2\alpha_1^2, \ldots, x_n^2 = t^2\alpha_n^2. \tag{2.6}$$

After summing the squares we get

$$R^2 = t^2 r^2, \quad R = |t|r, \tag{2.7}$$

with

$$R^2 = x_1^2 + \cdots + x_n^2, \quad r^2 = \alpha_1^2 + \cdots + \alpha_n^2. \tag{2.8}$$

Combining (2.5) with $x_{n+1} = \gamma(1-t)$ we have

$$x_{n+1} = R^2 = \gamma(1-t) = t^2 r^2, \quad t = 1 - \frac{R^2}{\gamma}. \tag{2.9}$$

Solving the quadratic equation $\gamma(1-t) = t^2 r^2$ for t we have

$$t = 1 - \frac{R^2}{\gamma} = \frac{2}{1 + \sqrt{1 + \frac{4r^2}{\gamma}}} \quad \text{if } 1 - \frac{R^2}{\gamma} \geqslant 0, \tag{2.10a}$$

$$t_- = 1 - \frac{R^2}{\gamma} = -\gamma \frac{1 + \sqrt{1 + \frac{4r^2}{\gamma}}}{2r^2} \quad \text{if } 1 - \frac{R^2}{\gamma} \leqslant 0. \tag{2.10b}$$

The formula (2.10a) enables us to write the coordinates of Z, on the parabolic bowl, in terms of the coordinates of z as follows:

$$x_1 = \frac{2\alpha_1}{1+\sqrt{1+\frac{4r^2}{\gamma}}}, \ldots, x_n = \frac{2\alpha_n}{1+\sqrt{1+\frac{4r^2}{\gamma}}},$$
$$x_{n+1} = \gamma(1-t) = \gamma\left[\frac{\sqrt{1+\frac{4r^2}{\gamma}}-1}{\sqrt{1+\frac{4r^2}{\gamma}}+1}\right]. \tag{2.11}$$

We also have via $R = |t|\, r$ in (2.7)

$$R^2 = \frac{2^2 r^2}{[1+\sqrt{1+\frac{4r^2}{\gamma}}]^2}, \quad R = \frac{2r}{1+\sqrt{1+\frac{4r^2}{\gamma}}}. \tag{2.12}$$

From the relation $t = 1 - \frac{R^2}{\gamma}$ we can obtain the inverse relations

$$\alpha_1 = \frac{x_1}{t} = \frac{x_1}{1-\frac{R^2}{\gamma}}, \ldots, \alpha_n = \frac{x_n}{t} = \frac{x_n}{1-\frac{R^2}{\gamma}}, \tag{2.13}$$

together with

$$r^2 = \frac{R^2}{t^2} = \frac{R^2}{\left(1-\frac{R^2}{\gamma}\right)^2}, \quad r = \frac{R}{\left|\left(1-\frac{R^2}{\gamma}\right)\right|}. \tag{2.14}$$

Notice that (2.11) holds only with Z on the parabolic bowl and of course $0 \leqslant t \leqslant 1$. Thus (2.11) together with (2.13) determine one of two possible bijections. On the other hand (2.13) detached from (2.11) could serve a dual purpose. With $1 - \frac{R^2}{\gamma} > 0$ we obtain $\mathbb{R}^n$ as the image of the parabolic bowl. With $t_- = 1 - \frac{R^2}{\gamma} < 0$ we obtain $\mathbb{R}^n$ as the image of the unbounded portion of the parabolic surface.

The relation (2.13), is central to our results about approximations via rational functions. The reason being is that $\alpha_1, \ldots, \alpha_n$ are rational functions of $x_1, \ldots, x_n$.

So far we have established a bijection from $\mathbb{R}^n$ to a subset of the parabolic surface (2.5). It is only natural now to take the limit in (2.11) as $r \to \infty$ in order to determine the definition of the correspondence between a point $z \in ID$ and a point Z on the parabolic surface. Consider, the relation

$$x_k = \frac{2\alpha_k}{1+\sqrt{1+\frac{4r^2}{\gamma}}} = \frac{2\alpha_k}{1+\sqrt{\frac{4r^2}{\gamma}\left(\frac{\gamma}{4r^2}+1\right)}} = \frac{2\alpha_k}{1+\frac{2r}{\sqrt{\gamma}}\left(\frac{\gamma}{4r^2}+1\right)^{\frac{1}{2}}}$$
$$= \left(\frac{\sqrt{\gamma}}{\frac{\sqrt{\gamma}}{2r}+\left[\frac{\gamma}{4r^2}+1\right]^{\frac{1}{2}}}\right)\left(\frac{\alpha_k}{r}\right), \quad k = 1, \ldots, n, \tag{2.15}$$

for $r > 0$. The coefficient of $\frac{\alpha_k}{r}$ in (2.15) tends to $\sqrt{\gamma}$ as $r \to \infty$. Notice also that $t \sim \frac{\sqrt{\gamma}}{r} \to 0$ as $r \to \infty$. It is now natural to match a point $\infty u \in ID$ with the point Z on the parabolic surface as follows:

$$x_1 = \sqrt{\gamma}u_1, \ldots, x_n = \sqrt{\gamma}u_n, \quad x_{n+1} = \gamma. \tag{2.16}$$

We add now the following definition:

Definition 2. The parabolic bowl is the set of points $Z = (x_1, \ldots, x_n, x_{n+1})$ defined by

$$\text{Parabolic bowl} := \{Z \mid x_1^2 + \cdots + x_n^2 \leqslant \gamma, 0 \leqslant x_{n+1} \leqslant \gamma\}. \tag{2.17}$$

Notice, that in the case that $n = 2$, our surface is obtained from revolving a parabola $x_3 = x_1^2$ about the x_3-axis. We then obtain a bijection of the ultra extended complex plane to a bowl-shaped set on the parabolic surface. Each point $\infty(\cos\,\theta, \sin\,\theta) \in ID$, $0 \leqslant \theta < 2\pi$ is matched with a point $Z = (\sqrt{\gamma}\cos\,\theta, \sqrt{\gamma}\sin\,\theta, \gamma)$ on a circle which is the boundary of the parabolic bowl.

We are ready now to summarize the discussion above by the following theorem.

Theorem 3. *Define the mapping*

$$x_1 = t\alpha_1, \ldots, x_n = t\alpha_n, \quad x_{n+1} = \gamma(1 - t) \tag{2.18}$$

with

$$t = \frac{2}{1 + \sqrt{1 + \frac{4r^2}{\gamma}}}, \quad r^2 = \alpha_1^2 + \cdots + \alpha_n^2, \quad z = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \tag{2.19}$$

and

$$x_1 = \sqrt{\gamma}u_1, \ldots, x_n = \sqrt{\gamma}u_n, \quad x_{n+1} = \gamma \tag{2.20}$$

for $z = \infty u$, $||u||^2 = 1$. *Then, this mapping establishes a bijection from* $\mathbb{R}^n \cup ID$ *to the closed set of the parabolic bowl. The inverse of this bijection is given by*

$$\alpha_1 = \frac{x_1}{1 - \frac{R^2}{\gamma}}, \ldots, \alpha_n = \frac{x_n}{1 - \frac{R^2}{\gamma}}. \tag{2.21}$$

Each α_k, $k = 1, 2, \ldots, n$, *is a rational function of the variables* $x_1, x_2, \ldots, x_n$.

We denote the bijection developed in formulas (2.18)–(2.21) *by* $Z = G(z)$.

We also have for $z \in \mathbb{R}^n$ *the mapping*

$$x_1 = -\gamma\frac{1 + \sqrt{1 + \frac{4r^2}{\gamma}}}{2r^2}\alpha_1, \ldots, x_n = -\gamma\frac{1 + \sqrt{1 + \frac{4r^2}{\gamma}}}{2r^2}\alpha_n,$$
$$x_{n+1} = \gamma\left(1 + \gamma\frac{1 + \sqrt{1 + \frac{4r^2}{\gamma}}}{2r^2}\right) \tag{2.22}$$

and for $z = \infty u$, $||u||^2 = 1$ *we have*

$$x_1 = -\sqrt{\gamma}u_1, \ldots, x_n = -\sqrt{\gamma}u_n, \quad x_{n+1} = \gamma. \tag{2.23}$$

The relations (2.22), (2.23), *together with the inverse relations given by* (2.13) *establish a bijection from* $\mathbb{R}^n \cup ID$ *to the subset of the parabolic surface given by*

$$\left\{ Z \mid Z = (x_1, \ldots, x_n, x_{n+1}),\ R^2 = x_1^2 + \cdots + x_n^2,\ \textit{and}\ 1 - \frac{R^2}{\gamma} \leqslant 0 \right\}. \tag{2.24}$$

3. The metric induced

In this section, we develop a metric on the set $\mathbb{R}^n \cup ID$ that is induced by the compactification of $\mathbb{R}^n$.

Supplement the notations of Section 1 by the following:

$$\hat{z} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_n), \quad \widehat{Z} = (\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{x}_{n+1}), \quad \hat{r}^2 = \sum_{j=1}^{n} \hat{\alpha}_j^2,$$

$$\widehat{R}^2 = \widehat{x}_1^2 +, \ldots, +\widehat{x}_n^2, \quad \hat{t} = \frac{2}{1 + \sqrt{1 + \frac{4\hat{r}^2}{\gamma}}}, \quad \hat{u} = (\hat{u}_{1,\ldots,}\hat{u}_n).$$

We define a metric $\chi(z, \hat{z})$ on $\mathbb{R}^n \cup ID$ by the Euclidean distance $||Z - \hat{Z}||$. Namely,

$$\chi(z, \hat{z}) = ||Z - \hat{Z}|| = ||G(z) - G(\hat{z})||. \tag{3.1}$$

We proceed now to express the distance between two points in $\mathbb{R}^n \cup ID$ in terms of their coordinates. By the definition of the Euclidean distance we have

$$\begin{aligned} ||Z - \hat{Z}||^2 &= \sum_{j=1}^{n} (\hat{x}_j - x_j)^2 + (\hat{x}_{n+1} - x_{n+1})^2 \\ &= \sum_{j=1}^{n} \hat{x}_j^2 + \sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n} 2x_j\hat{x}_j + (\hat{x}_{n+1} - x_{n+1})^2. \end{aligned} \tag{3.2}$$

Substitute in (3.2), $\hat{x}_{n+1} = \sum_{j=1}^{n} \hat{x}_j^2, \quad x_{n+1} = \sum_{j=1}^{n} x_j^2$. Then,

$$||Z - \hat{Z}||^2 = \hat{x}_{n+1} + x_{n+1} + (\hat{x}_{n+1} - x_{n+1})^2 - \sum_{j=1}^{n} 2x_j\hat{x}_j. \tag{3.3}$$

Now substitute in (3.3); $\hat{x}_{n+1} = \gamma(1 - \hat{t})$, $x_{n+1} = \gamma(1 - t)$, $\hat{x}_j = \hat{t}\hat{\alpha}_j$, $x_j = t\alpha_j$, $j = 1, \ldots, n$, and obtain

$$||Z - \hat{Z}||^2 = \gamma^2[\hat{t} - t]^2 + \gamma[2 - (t + \hat{t})] - 2\hat{t}t \sum_{j=1}^{n} \alpha_j\hat{\alpha}_j. \tag{3.4}$$

We utilize the Euclidean distance in $\mathbb{R}^n$, namely,

$$D^2 = \sum_{j=1}^{n} (\hat{\alpha}_j - \alpha_j)^2 = \sum_{j=1}^{n} \hat{\alpha}_j^2 + \sum_{j=1}^{n} \alpha_j^2 - 2\sum_{j=1}^{n} \alpha_j \hat{\alpha}_j$$
$$= \hat{r}^2 + r^2 - 2\sum_{j=1}^{n} \alpha_j \hat{\alpha}_j \tag{3.5}$$

and we substitute in (3.5)

$$-2\sum_{j=1}^{n} \alpha_j \hat{\alpha}_j = D^2 - \hat{r}^2 - r^2. \tag{3.6}$$

Thus we obtain

$$||Z - \hat{Z}||^2 = \gamma^2[\hat{t} - t]^2 + \gamma[2 - (t + \hat{t})] + t\hat{t}[D^2 - \hat{r}^2 - r^2]$$
$$= \gamma^2[\hat{t} - t]^2 + \gamma[2 - (t + \hat{t})] + t\hat{t}\left[D^2 - \frac{\gamma(1-t)}{t^2} - \frac{\gamma(1-\hat{t})}{\hat{t}^2}\right],$$

by virtue of $r^2 = \frac{\gamma(1-t)}{t^2}$ and $\hat{r}^2 = \frac{\gamma(1-\hat{t})}{\hat{t}^2}$.

We now rewrite the right-hand side of (3.4) as follows:

$$||Z - \hat{Z}||^2 = t\hat{t}\left[D^2 - \frac{\gamma(1-t)}{t^2} - \frac{\gamma(1-\hat{t})}{\hat{t}^2} + \frac{\gamma^2[\hat{t} - t]^2}{t\hat{t}} + \frac{\gamma[2 - (t + \hat{t})]}{t\hat{t}}\right].$$

A short calculation reveals that,

$$||Z - \hat{Z}||^2 = t\hat{t}\left[D^2 + \frac{\gamma^2[\hat{t} - t]^2}{t\hat{t}} - \gamma\left(\frac{1}{\hat{t}} - \frac{1}{t}\right)^2\right]$$
$$= t\hat{t}\left\{D^2 - \gamma\left(\frac{1}{\hat{t}} - \frac{1}{t}\right)^2 [1 - \gamma t\hat{t}]\right\}. \tag{3.7}$$

Finally, we have an expression of the metric in terms of the coordinates of z and $\hat{z}$,

$$\chi(z, \hat{z}) = ||G(\hat{z}) - G(z)|| = \sqrt{t\hat{t}D^2 + \gamma^2[\hat{t} - t]^2 - \frac{\gamma(t - \hat{t})^2}{t\hat{t}}}$$
$$= \sqrt{t\hat{t}\left\{D^2 - \gamma\left(\frac{1}{\hat{t}} - \frac{1}{t}\right)^2 [1 - \gamma t\hat{t}]\right\}}. \tag{3.8}$$

This with $\hat{t} = \frac{2}{1+\sqrt{1+\frac{4\hat{r}^2}{\gamma}}}$ and $t = \frac{2}{1+\sqrt{1+\frac{4r^2}{\gamma}}}$.

If one of the points z or $\hat{z}$ is in the ideal set ID, say $z = \infty u$, $||u|| = 1$, then we obtain directly from (3.2)

$$\chi(\infty u, \hat{z}) = \sqrt{\sum_{j=1}^{n} (\hat{t}\hat{\alpha}_j - \sqrt{\gamma} u_j)^2 + \gamma^2 \hat{t}^2}. \tag{3.9}$$

This, by virtue of the fact that $t \sim \frac{\sqrt{\gamma}}{r} \to 0$ as $r \to \infty$.
Moreover, with $z = \infty u$, $\|u\| = 1$ and $\hat{z} = \infty \hat{u}$, $\|\hat{u}\| = 1$ we have

$$\chi(\infty u, \infty \hat{u}) = \sqrt{\gamma \sum_{j=1}^{n} (\hat{u}_j - u_j)^2}. \tag{3.10}$$

A lengthy but straightforward calculation yields the metric expressed explicitly in terms of D^2 and the radial distances r^2 and $\hat{r}^2$. Videlicet,

$$\begin{aligned}
\chi^2(z, \hat{z}) = ||Z - \hat{Z}||^2 &= \frac{2^2}{[1+\sqrt{1+\frac{4r^2}{\gamma}}][1+\sqrt{1+\frac{4\hat{r}^2}{\gamma}}]} \\
&\quad \times \left\{ D^2 - \frac{\gamma}{2^2}\left[\sqrt{1+\frac{4r^2}{\gamma}} - \sqrt{1+\frac{4\hat{r}^2}{\gamma}}\right]^2 \right. \\
&\quad \left. \times \left(1 - \frac{2^2\gamma}{[1+\sqrt{1+\frac{4r^2}{\gamma}}][1+\sqrt{1+\frac{4\hat{r}^2}{\gamma}}]}\right)\right\} \\
&= \frac{2^2}{[1+\sqrt{1+\frac{4r^2}{\gamma}}][1+\sqrt{1+\frac{4\hat{r}^2}{\gamma}}]} \\
&\quad \times \left\{ D^2 - \frac{2^2(r^2-\hat{r}^2)^2}{\gamma[\sqrt{1+\frac{4r^2}{\gamma}} + \sqrt{1+\frac{4\hat{r}^2}{\gamma}}]^2} \right. \\
&\quad \left. \times \left(1 - \frac{2^2\gamma}{[1+\sqrt{1+\frac{4r^2}{\gamma}}][1+\sqrt{1+\frac{4\hat{r}^2}{\gamma}}]}\right)\right\}.
\end{aligned} \tag{3.11}$$

Since $\sqrt{1+\frac{4r^2}{\gamma}} - \sqrt{1+\frac{4\hat{r}^2}{\gamma}} = \frac{2^2(r^2-\hat{r}^2)}{\gamma[\sqrt{1+\frac{4r^2}{\gamma}}+\sqrt{1+\frac{4\hat{r}^2}{\gamma}}]}$.

Let us calculate in the case $n = 2$ the parabolic distance between two points in the ideal set ID. A straightforward calculation reveals that

$$\begin{aligned}
\chi(\infty(\cos\,\theta, \sin\,\theta), \infty(\cos\,\psi, \sin\,\psi)) &= \sqrt{2\gamma[1-\cos(\theta-\psi)]} \\
&= 2\sqrt{\gamma}\left|\sin\,\frac{(\theta-\psi)}{2}\right|.
\end{aligned}$$

In particular, if $n = 1$, the parabolic distance between $+\infty$ and $-\infty$ is $2\sqrt{\gamma}$.
Although the parameter γ adds a coordinate of freedom to our setting, we will take $\gamma = 1$ in the remaining sections.

4. General and rational approximations

In this section, we point out how to employ general type approximations in the service of members of the family $CAI[a,b]$. We also derive an extended Weierstrass theorem and an extended Fourier-type theorem. We observe that $f(s) \in CAI[a,b]$ implies that $\frac{2f(s)}{1+\sqrt{1+4f^2(s)}} \in C[a,b]$.

Hence, any approximation sequence, that is available in the literature, that approximates the continuous function $\frac{2f(s)}{1+\sqrt{1+4f^2(s)}} \in C[a,b]$ in the supremum norm, is an approximation tool for $f(s) \in CAI[a,b]$. The special nature of the parabolic compactification becomes then crucial in obtaining rational approximations.

The following proposition employs quite general approximants.

Proposition 4. *Consider $f(s) \in CAI[a,b]$. Define the sequence $\widetilde{A} := \widetilde{A}(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s)$ that is given by*

$$\widetilde{A} := \frac{A(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s)}{1 - A^2(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s)}. \tag{4.1}$$

Assume that uniformly on $[a,b]$,

$$\lim_{n\to\infty} \sup \left| A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right) - \frac{2f(s)}{1+\sqrt{1+4f^2(s)}} \right| = 0. \tag{4.2}$$

Then,

(i) *on every compact subset I of $[a,b]$ that excludes points $\hat{s}$, where $f(\hat{s})$ is unbounded we have*

$$\lim_{n\to\infty} \sup_{s\in I} \left| f(s) - \widetilde{A}\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right) \right| = 0. \tag{4.3}$$

(ii) *The sequence $\frac{A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)}{\left|1-A^2\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)\right|}$ converges uniformly on $[a,b]$ to $f(s)$ in the parabolic metric.*

Proof. A proof of (i) requires to show that a small neighborhood on the parabolic bowl must be the image of a small neighborhood of $\mathbb{R}^n$ if the neighborhood in $\mathbb{R}^n$ is confined to a compact subset of $\mathbb{R}^n$. This follows by scrutinizing the relation between $\chi^2(z,\hat{z})$ and $D^2 = \| z - \hat{z} \|^2$ in formulas (3.8), (3.11) and the related quantities t and $\hat{t}$. To that end notice that the function t is a monotone decreasing function of r^2 since

$$\frac{\partial t}{\partial (r^2)} = \frac{-4}{\left(1 + \frac{4r^2}{\gamma}\right)^{1/2} \left[1 + \sqrt{1 + \frac{4r^2}{\gamma}}\right]^2} < 0. \tag{4.4}$$

Therefore, $t(0) = 1$, $t(\infty) = 0$ and $0 \leqslant t \leqslant 1$. Moreover, if $r^2 \leqslant \zeta^2 < \infty$, then $0 < \delta \leqslant t \leqslant 1$ with $\delta = \frac{2}{1+\sqrt{1+4\zeta^2/\gamma}}$. Recall that

$$t = \frac{2}{1+\sqrt{1+\frac{4r^2}{\gamma}}} = 1 - \frac{R^2}{\gamma}, \quad R^2 = t^2 r^2 = \left(1 - \frac{R^2}{\gamma}\right)^2 r^2,$$

$$r^2 = \frac{R^2}{(1-\frac{R^2}{\gamma})^2}, \quad x_{n+1} = R^2 = \gamma(1-t).$$

From $R^2 = \gamma(1-t)$ and t a decreasing function of r^2 we conclude that $r^2 \leqslant \zeta^2 < \infty$ implies

$$R^2 \leqslant \gamma(1 - t(\zeta^2)) < \gamma, \quad 1 - \frac{R^2}{\gamma} \geqslant t(\zeta^2) > 0. \tag{4.5}$$

From (3.8) we have

$$t\hat{t}D^2 = \chi^2(z,\hat{z}) - \gamma^2[\hat{t}-t]^2 + \frac{\gamma(t-\hat{t})^2}{t\hat{t}} = \chi^2(z,\hat{z}) + \gamma(t-\hat{t})^2\left[(t\hat{t})^{-1} - \gamma\right]. \tag{4.6}$$

We intend to find a bound on the factor $\gamma(t-\hat{t})^2$ on the right-hand side of (4.6), in terms of $\chi^2(z,\hat{z})$. We have

$$\gamma(t-\hat{t})^2 = \gamma\left[\left(1 - \frac{R^2}{\gamma}\right) - \left(1 - \frac{\hat{R}^2}{\gamma}\right)\right]^2 = \gamma^{-1}(\hat{R}-R)^2(\hat{R}+R)^2. \tag{4.7}$$

From (3.2) we conclude that

$$\begin{aligned} \chi^2(z,\hat{z}) &= R^2 + \hat{R}^2 - 2\sum_{j=1}^{n} x_j\hat{x}_j + (\hat{R}-R)^2 \\ &\geqslant R^2 + \hat{R}^2 - 2R\hat{R} + (\hat{R}^2 - R^2)^2 \\ &= (\hat{R}-R)^2\left[1 + (\hat{R}+R)^2\right] \geqslant (\hat{R}-R)^2. \end{aligned} \tag{4.8}$$

This is so by the Cauchy–Schwarz inequality, as

$$\left|\sum_{j=1}^{n} x_j\hat{x}_j\right| \leqslant \left[\sum_{j=1}^{n} x_j^2\right]^{\frac{1}{2}} \left[\sum_{j=1}^{n} \hat{x}_j^2\right]^{\frac{1}{2}} = R\hat{R}. \tag{4.9}$$

Hence,

$$\mid R - \hat{R} \mid \leqslant \chi(z,\hat{z}). \tag{4.10}$$

Utilize in (4.6) the conclusions (4.7) and (4.10) and obtain

$$\left|t\hat{t}\right| D^2 \leqslant \left[1 + \gamma^{-1}(\hat{R}+R)^2\left(\left|t\hat{t}\right|^{-1} + \gamma\right)\right]\chi^2(z,\hat{z}). \tag{4.11}$$

Finally we conclude from (4.11) the relation

$$
\begin{aligned}
D^2 &\leqslant \left|t\hat{t}\right|^{-1}\left[1+(\hat{R}+R)^2\left(\left|\gamma t\hat{t}\right|^{-1}+1\right)\right]\chi^2(z,\hat{z})\\
&\leqslant \left|\left(1-\frac{R^2}{\gamma}\right)\left(1-\frac{\hat{R}^2}{\gamma}\right)\right|^{-1}\\
&\quad\times\left[1+(\hat{R}+R)^2\left(\left|\gamma\left(1-\frac{R^2}{\gamma}\right)\left(1-\frac{\hat{R}^2}{\gamma}\right)\right|^{-1}+1\right)\right]\chi^2(z,\hat{z}). \qquad (4.12a)
\end{aligned}
$$

Recall the relation (4.5), where $r^2 \leqslant \zeta^2 < \infty$ and $\widehat{r}^2 \leqslant \zeta^2 < \infty$, imply $1-\frac{R^2}{\gamma} \geqslant t(\zeta^2) > 0$ and $1-\frac{\widehat{R}^2}{\gamma} \geqslant t(\zeta^2) > 0$ and obtain

$$
D^2 \leqslant \left[t(\zeta^2)\right]^{-2}\left\{1+\left[2\zeta t(\zeta^2)\right]^2\left(\gamma^{-1}\left[t(\zeta^2)\right]^{-2}+1\right)\right\}\chi^2(z,\hat{z}). \qquad (4.12b)
$$

Hence, if $\chi(z,\hat{z})$ tends to zero so does D^2 for $r^2, \hat{r}^2 \leqslant \zeta^2 < \infty$.

It is now possible to conclude the proof of (i) by pointing out the following relations. Put in (4.12) $\gamma = 1$ and put

$$
z = f(s), \quad Z = G(z) = \frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, \qquad (4.13a)
$$

$$
\hat{Z} = A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right), \quad \hat{z} = \frac{A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)}{1-A^2\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)}, \qquad (4.13b)
$$

$$
r = |f(s)|, \quad R = \left|\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}\right|, \qquad (4.13c)
$$

$$
\hat{R} = \left|A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)\right|, \quad \hat{r} = \left|\frac{A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)}{1-A^2\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)}\right|. \qquad (4.13d)
$$

The relation (4.2) implies that on every compact subset I of $[a, b]$ that excludes points $\hat{s}$ where $f(\hat{s})$ is unbounded, there exists $\zeta^2 < \infty$, for n large enough such that the (equal) quantities $t^{-1}(\zeta^2), \hat{t}^{-1}(\zeta^2)$ are bounded.

Moreover, the fact that on I we have $-1+\theta \leqslant \frac{2f(s)}{1+\sqrt{1+4f^2(s)}} \leqslant 1-\theta$, for some positive fixed θ guarantees that there exists a fixed $\widehat{\theta} > 0$, such that on I we have $1-A^2\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right) \geqslant \widehat{\theta} > 0$ for n large enough. The inequality (4.12b) yields then the desired result (4.2).

Let us sketch the main features of the proof of (ii) in order to clarify the nature of the technicalities that will follow. Unfortunately, the sequence $\widetilde{A}\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)$ does not converge to $f(s)$ in the parabolic metric.

This is due to the fact that $t\left[\left(\frac{x}{1-x^2}\right)^2\right]\frac{x}{1-x^2}$ is not identical to x for $1-x^2 < 0$. We keep in mind that we identify x with $A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)$ and that we cannot guarantee that $1 - A^2\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right) \geqslant 0$. However, it can easily be verified that

$$t\left[\left(\frac{x}{\left|1-x^2\right|}\right)^2\right]\frac{x}{\left|1-x^2\right|} = \begin{cases} x & \text{if } 1-x^2 \geqslant 0, \\ x^{-1} & \text{if } 1-x^2 \leqslant 0. \end{cases} \tag{4.14}$$

For the bulk of the values of s on $[a, b]$, we have $1 - A^2\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right) \geqslant 0$ for n large enough. This can be deduced from (4.2). If $1 - A^2\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right) \leqslant 0$, then (4.2) tells us that $A^2\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)$ must be close to 1. Then the three quantities $A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)$, $A^{-1}\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)$, and $\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}$ are indeed close to each other. These arguments motivate the technicalities below.

The relation (4.2) implies that for every $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for $n > N(\varepsilon)$ we have, on $[a, b]$

$$-\varepsilon < \frac{2f(s)}{1+\sqrt{1+4f^2(s)}} - A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right) < \varepsilon. \tag{4.15}$$

We identify on the interval $[a, b]$, for each fixed θ, $0 < \theta < 1$, three subsets I_n, $I_{+1n\theta}$ and $I_{-1n\theta}$ as follows:

$$I_n := \left\{s \,|\, s \in [a, b],\ -1 \leqslant A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right) \leqslant 1\right\}, \tag{4.16a}$$

$$I_{-1n\theta} := \left\{s \,|\, s \in [a, b],\ -1-\theta \leqslant A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right) \leqslant -1\right\}, \tag{4.16b}$$

$$I_{+1n\theta} := \left\{s \,|\, s \in [a, b],\ 1 \leqslant A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right) \leqslant 1+\theta\right\}. \tag{4.16c}$$

Evidently $[a, b] = I_n \cup I_{+1n\theta} \cup I_{-1n\theta}$ if ε is small enough and n is large enough. From (4.16c) we derive the inequality

$$-1 \leqslant -\frac{1}{A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)} \leqslant -\frac{1}{1+\theta}. \tag{4.17}$$

Its combination with (4.16c) results in the inequality

$$0 \leqslant A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right) - \frac{1}{A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)} \leqslant \frac{\theta(2+\theta)}{1+\theta}. \tag{4.18}$$

The combination of (4.15) and (4.18) implies that on $I_{+1n\theta}$ we have for $n > N(\varepsilon)$

$$-\varepsilon < \frac{2f(s)}{1+\sqrt{1+4f^2(s)}} - \frac{1}{A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)} < \varepsilon + \frac{\theta(2+\theta)}{1+\theta}. \tag{4.19}$$

In a similar manner we have on $I_{-1n\theta}$

$$-\varepsilon - \frac{\theta}{1+\theta} < \frac{2f(s)}{1+\sqrt{1+4f^2(s)}} - \frac{1}{A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)} < \varepsilon. \tag{4.20}$$

Evidently the relation (4.15) holds on I_n. Choose now $\theta = \varepsilon < 1$. Consider the set of bounds on the right-hand side and on the left-hand side of (4.15), (4.19) and (4.20). Then $\max\left\{\varepsilon, \varepsilon + \frac{\varepsilon(2+\varepsilon)}{1+\varepsilon}, \varepsilon + \frac{\varepsilon}{1+\varepsilon}\right\} < 4\varepsilon$. By virtue of (4.14), (4.15), (4.19) and (4.20) we have that the sequence $\frac{A\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)}{\left|1-A^2\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)\right|}$ converges uniformly to $f(s)$ on $[a,\ b]$ in the parabolic metric. □

The fact that on every compact subinterval I of $[a, b]$, that excludes singularities of $f(s)$, we have $\lim_{n\to\infty} \sup_{s\in I} |f(s) - \hat{A}(f, n, s)| = 0$, has a few useful implications. The first implication is that if indeed $\left|f(\hat{s})\right| = \infty$ for some $\hat{s} \in [a, b]$ then $\hat{A}(f, n, s)$ must become unbounded for a sequence $s_n \to \hat{s}$, $s_n \in [a, b]$ as $n \to \infty$. This is an indication that if for n large enough, $\hat{A}(f, n, s)$ possess singularities $\hat{s}_n$, then these singularities should cluster around $\hat{s}$. Naturally, these singularities should coincide with the roots of $1 = A^2\left(\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}, n, s\right)$. It is plausible that an a priori knowledge of the nature of smoothness and singularities of $f(s)$ could yield more precise information on the location and nature of singularities of $\hat{A}(f, n, s)$ whenever they exist. This is born out by the theory of Pade approximations. Compare e.g. with [2,10]. If $f(s)$ is meromorphic in a disk $|s| < R$, then the poles of the Pade approximants converge to the poles of $f(s)$.

It is possible now to obtain the analogs of Weierstrass' theorem and Fourier's-type theorem for functions $f(s) \in CAI[a,b]$. By Weierstrass' theorem there exist Polynomials $P_n(s)$ such that

$$\lim_{n\to\infty} \sup \left| \frac{2f(s)}{1+\sqrt{1+4f^2(s)}} - P_n(s) \right| = 0$$

uniformly on $[a,b]$. Consequently, $\hat{P}_n(s) := \frac{P_n(s)}{|1-P_n^2(s)|}$ is a rational function which approximates $f(s)$ uniformly on $[a,b]$ in the parabolic metric.

Let $f(s) \in CAI[-l,l]$, $l > 0$, be a periodic function such that $f(s+2l) = f(s)$, $-\infty < s < \infty$. Then, under various assumptions on the smoothness of $f(s)$, e.g. [11,13], the function $\frac{2f(s)}{1+\sqrt{1+\frac{4f^2(s)}{\gamma}}}$ possesses a converging Fourier series. The Fourier series are approximated by the partial sums $F_n(s)$, $n = 0,1,2,\ldots$. These are polynomials of degree n in $\cos s$ and $\sin s$. Consequently, the sequence of quotients of trigonometric polynomials $\frac{F_n(s)}{|1-F_n^2(s)|}$, converges in the parabolic metric to $f(s)$.

5. Unboundedly compatible approximations

The approximations $\hat{A}(f,n,s)$ discussed in the previous section do not guarantee that $\hat{A}(f,n,s) \neq +\infty$ or $-\infty$ whenever $f(s) \neq +\infty$ or $-\infty$, respectively, for all $n = 1,2,\ldots$. We are only guaranteed that for n large enough, $\hat{A}(f,n,s) \neq +\infty$ or $-\infty$, on every compact subset of a finite interval $[a,b]$ whenever $f(s) \neq \infty$ or $-\infty$, respectively. A natural question arises then. How large should n be in order to prevent a catastrophic occurrence where $\hat{A}(f,n,s) = \infty$ or $-\infty$ for all values s where $f(s) \neq +\infty$ or $-\infty$? An answer to this is not readily available in the results of the previous section. One could use trial and error in the process of practical implementation of numerical schemes in order to avoid the mentioned catastrophical occurrences. However, it is preferable to look for means that will advance our knowledge in these matters. In the process of providing such means we will employ the Bernstein's polynomials.

First we need some definitions.

Definition 5. Let $g(s) \in C([0,1])$, the nth Bernstein's polynomial is given by

$$B_n(s) := \sum_{j=0}^{n} C_j^n s^j (1-s)^{n-j} g\left(\frac{j}{n}\right), \quad \text{for all } n \geqslant 1, \text{ and, } s \in [0,1], \tag{5.1}$$

where $C_j^n = \frac{n!}{j!(n-j)!}$ are the binomial coefficients.

Definition 6. Let $f \in CAI\ [a,\ b]$. We define for each integer $n \geqslant 1$ the number of equidistant points $s = a + \frac{j(b-a)}{n}$, $j = 0,\ldots,n$, where $f(s) = +\infty$, and the number of equidistant points $s = a + \frac{j(b-a)}{n}$, $j = 0,\ldots,n$, where $f(s) = -\infty$, respectively, as $M(n,+\infty)$ and

$M(n, -\infty)$, that is to say $M(n, +\infty) := card\left(\left\{j,\ j = 0, \ldots, n, : f(a + \frac{j(b-a)}{n}) = \infty\right\}\right)$ and $M(n, -\infty) := card\left(\left\{j,\ j = 0, \ldots, n, : f(a + \frac{j(b-a)}{n}) = -\infty\right\}\right)$.

Let $V(s)$ be a vector function such that $V(s) \in CAI[a, b]$. We denote for each integer $n \geqslant 1$ by $K(n, \infty)$, the number of equidistant points $s = a + \frac{j(b-a)}{n}$, $j = 0, \ldots, n$, where $\left\| V(a + \frac{j(b-a)}{n}) \right\|^2 = \sum_{i=1}^{n} v_i^2(a + \frac{j(b-a)}{n}) = \infty$. That is to say $K(n, \infty) := card\left(\left\{j,\ j = 0, \ldots, n, : \left\| V(a + \frac{j(b-a)}{n}) \right\| = \infty\right\}\right)$.

Definition 7. Let $\hat{A}(f, n, s)$, $n = 1, 2, \ldots$ be *a sequence of approximants to a function* $f(s) \in CAI[a, b]$. We say that $\hat{A}(f, n, s)$ is N unboundedly compatible with $f(s)$ if $|\hat{A}(f, n, s)| \neq \infty$ whenever $|f(s)| \neq \infty$ for $n > N$.

Let $W(V, n, s)$ $n = 1, 2, \ldots$ be *a sequence of* vector approximants to a vector function $V(s) \in CAI[a, b]$. We say that $W(V, n, s)$ is N unboundedly compatible with the vector function $V(s)$ if $\| W(V, n, s) \| \neq \infty$ whenever $\| V(s) \| \neq \infty$ for $n > N$.

The reason that Bernstein's polynomials are a desired tool is revealed in the following theorem.

Theorem 8. *If $f \in CAI[0, 1]$ is such that $M(n, +\infty) < n + 1$ and $M(n, -\infty) < n + 1$ for $n > N$, N a fixed integer, then*

(i) *The sequence*

$$\hat{A}(f, n, s) := \frac{B_n(s)}{1 - B_n^2(s)}, \quad n \geqslant 1 \tag{5.2a}$$

is N unboundedly compatible with $f(s)$ and

$$B_n(s) = \sum_{j=0}^{n} \frac{2f(j/n)}{1 + \sqrt{1 + 4f^2(j/n)}} C_j^n s^j (1 - s)^{n-j}. \tag{5.2b}$$

(ii) *Furthermore, $\hat{A}(f, n, s)$ converge uniformly on $[0, 1]$ to $f(s)$ as $n \to \infty$, in the parabolic metric.*

(iii) *On every closed subset $I \subset [0, 1]$, such that $f(s) \in C(I)$ we have*

$$\lim_{n \to \infty} \sup_{s \in I} \left| f(s) - \frac{B_n(s)}{1 - B_n^2(s)} \right| = 0. \tag{5.3}$$

Notice that the results are easily transferable to $[a,\ b]$ by means of the linear transformation $y = \frac{x-a}{b-a}$ that converts $[a,\ b]$ into $[0,\ 1]$.

Proof. We observe that

$$1 \equiv [s + (1 - s)]^n = \sum_{j=0}^{n} C_j^n s^j (1 - s)^{n-j} \tag{5.4}$$

and that

$$-1 \leqslant \frac{2f(s)}{1+\sqrt{1+4f^2(s)}} \leqslant 1. \tag{5.5}$$

Hence,

$$-1 \leqslant B_n(s) = \sum_{j=0}^{n} \frac{2f(j/n)}{1+\sqrt{1+4f^2(j/n)}} C_j^n s^j (1-s)^{n-j} \leqslant 1. \tag{5.6}$$

Obviously, $\hat{A}(f,n,s) = +\infty$ iff $B_n(s) = 1$ or iff

$$\sum_{j=0}^{n} \left[1 - \frac{2f(j/n)}{1+\sqrt{1+4f^2(j/n)}}\right] C_j^n s^j (1-s)^{n-j} = 0. \tag{5.7}$$

Each term in the sum given on the left-hand side of Eq. (5.7) is non-negative. Hence (5.7) can be materialized iff for each j, $j = 0, 1, 2, \ldots, n,$ we have

$$\left[1 - \frac{2f(j/n)}{1+\sqrt{1+4f^2(j/n)}}\right] C_j^n s^j (1-s)^{n-j} = 0. \tag{5.8}$$

There are three cases to be considered. The case $0 < s < 1$, the case $s = 0$ and the case $s = 1$. If $0 < s < 1$ then $C_j^n s^j (1-s)^{n-j} \neq 0$ for all $j = 0, 1, 2, \ldots, n, n = 1, 2, \ldots$. This requires for fixed n

$$\frac{2f(j/n)}{1+\sqrt{1+4f^2(j/n)}} = 1, \quad j = 0, 1, 2, \ldots, n, \quad n = 1, 2, \ldots \tag{5.9}$$

or

$$f\left(\frac{j}{n}\right) = +\infty, \quad j = 0, 1, 2, \ldots, n. \tag{5.10}$$

Hence (5.10) requires $(n+1)$ equidistant points where (5.10) is satisfied.

This is impossible if $n > N$. If $s = 0$, then $\frac{2f(0)}{1+\sqrt{1+4f^2(0)}} = 1$ iff $f(0) = +\infty$. Similarly, $\frac{2f(1)}{1+\sqrt{1+f^2(1)}} = 1$ iff $f(1) = +\infty$. The arguments are similar for the case $-\infty$. $\square$

It goes without saying that if $f(s)$ possesses a finite number N of discontinuities then $\hat{A}(f,n,s)$ is N *unboundedly compatible with* $f(s)$. Notice that Theorem 8 allows $f(s)$ to possess infinitely many points of discontinuity that are not equally spaced. For example consider the function $f(s) = s^{-1}\csc^2\frac{\pi}{s}$. Evidently, $f(s) \in CAI[0, 1]$ with points of discontinuity at $s = 0$ and $\frac{1}{m}$, m a positive integer. We claim that $\hat{A}(f,n,s)$ is $N = 2$ *unboundedly compatible with* $f(s)$. Assume by contradiction that this is false. Then $s = \frac{1}{m_j} = \frac{j}{n}$ and $j = 0, 1, 2, \ldots, n-1, n$, and m_j some positive integers. This, for each fixed integer n in an infinite sequence of values of n. This is so iff for fixed n we have $n = jm_j$, and $j = 0, 1, \ldots, n-1, n$ that implies that n is divisible by $(n-1)$. This is impossible for $n > 2$ and the conclusion follows.

Let us proceed with an informal discussion of the approximations of periodic functions $f(s) = f(s+2\pi)$ and $f \in CAI[-\pi, \pi]$. To this end we construct the coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{2f(s)\,\cos ns}{1+\sqrt{1+4f^2(s)}}\,ds, \tag{5.11}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{2f(s)\,\sin ns}{1+\sqrt{1+4f^2(s)}}\,ds, \tag{5.12}$$

in order to obtain the partial sums

$$F_n(s) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos ks + b_k \sin ks], \quad n = 0, 1, 2, \dots . \tag{5.13}$$

We generate the Cesaro sums

$$\sigma_0(s) = \frac{F_0(s)}{1}, \quad \sigma_n(s) = \frac{F_0(s) + F_1(s) + \cdots + F_n(s)}{n+1}. \tag{5.14}$$

Evidently, $\sigma_n(s)$ are "trigonometric sums" of degree n, $n = 0, 1, 2, \dots$. Namely, $\sigma_n(s)$ are polynomials of degree n in the two variables cos s and sin s. By Fejer's theorem,

$$\sigma_n(s) = \int_{-\pi}^{\pi} K(s,\eta,n)\frac{2f(\eta)}{1+\sqrt{1+4f^2(\eta)}}\,d\eta \tag{5.15a}$$

with $K(s,\eta,n)$ the positive Kernel that satisfies

$$K(s,\eta,n) = \frac{1}{2\pi(n+1)}\left[\frac{\sin\frac{(n+1)(\eta-s)}{2}}{\sin\frac{(\eta-s)}{2}}\right]^2. \tag{5.15b}$$

See e.g. [7]. Consequently, $m \leqslant \frac{2f(s)}{1+\sqrt{1+4f^2(s)}} \leqslant M$ implies $m \leqslant \sigma_n(s) \leqslant M$.

Moreover, if $M - m > 0$ and M and m are the absolute minimum and absolute maximum values of $\frac{2f(s)}{1+\sqrt{1+4f^2(s)}}$ on $(-\infty, \infty)$, respectively, then $m < \sigma_n(s) < M$ for $-\infty < s < \infty$. To prove the above statement we notice that $M - m > 0$ implies that there exists a point $\eta_0 \in [-\pi, \pi]$ such that

$$M - \frac{2f(\eta_0)}{1+\sqrt{1+4f^2(\eta_0)}} > 0. \tag{5.16}$$

By continuity we have then $\int_{-\pi}^{\pi}[M - \frac{2f(s)}{1+\sqrt{1+4f^2(s)}}]K(s,\eta,n)\,d\eta > 0$ or that

$$\sigma_n(s) = \int_{-\pi}^{\pi} K(s,\eta,n)\frac{2f(\eta)}{1+\sqrt{1+4f^2(\eta)}}\,d\eta < M\int_{-\pi}^{\pi} K(s,\eta,n)\,d\eta = M. \tag{5.17}$$

In a similar manner we have $m < \sigma_n(s)$ for $-\infty < s < \infty$. Since $-1 \leqslant \frac{2f(s)}{1+\sqrt{1+4f^2(s)}} \leqslant 1$, then $-1 < \sigma_n(s) < 1$, $n = 0, 1, 2, \dots$ for $s \in (-\infty, \infty)$ if $f(s)$ is not the constant ∞ or $-\infty$.

We can summarize formally the above discussion in the following theorem:

Theorem 9. *Let* $f(s) \in CAI[-\pi, \pi]$ *and* $f(s+2\pi) = f(s)$. *Then,*

(i) *there exists quotients of "trigonometric sums"* $\frac{\sigma_n(s)}{1-\sigma_n^2(s)}$ *that converge uniformly to* $f(s)$ *on* $(-\infty, \infty)$, *as* $n \to \infty$, *in the parabolic metric.*

(ii) *On every subset* $I \subset (-\infty, \infty)$, *that excludes points where* $|f(s)| = \infty$ *we have uniformly*

$$\lim_{n\to\infty} \sup \left| f(s) - \frac{\sigma_n(s)}{1-\sigma_n^2(s)} \right| = 0. \tag{5.18}$$

(iii) *Moreover,* $1 - \sigma_n^2(s) > 0$, *for* $n = 1, 2, \ldots, \ -\infty < s < \infty$.

The approximation of vector functions $V = V(s) = \langle v_1(s), v_2(s), \ldots, v_m(s) \rangle$, $v_k(s) \in CAI[a, b]$, $k = 1, 2, \ldots, m$, may be done in two ways. In the first way, each component $v_k(s)$ could be approximated by a rational function $\frac{B_{kn}(s)}{1-B_{kn}^2(s)}$, with $B_{kn}(s)$ the Bernstein polynomials utilized in Theorem 8. This involves the simultaneous evaluation of m different denominators and m different square roots in the expressions $\frac{2v_{kn}(s)}{1+\sqrt{1+4v_{kn}^2(s)}}$, $k = 1, 2, \ldots, m$. It is possible to obviate the necessity to calculate the m different denominators and m different square roots by utilizing the parabolic compactification of $\mathbb{R}^n$ for the vector function $V(s)$. Thus, making our computation more efficient. Moreover, the approximations could be made unboundedly compatible with $V(s)$. This is given in the next proposition.

Proposition 10. *Let* $V(s) \in CAI[0, 1]$. *Let* $K(n, \infty) < n + 1$ *for* $n > N$. *Then,*

(i) *the vector* $W = W(V, n, s)$ *defined by*

$$W(V, n, s) = \frac{1}{1 - \sum_{k=1}^{m} B_{kn}^2(s)} \langle B_{1n}(s), B_{2n}(s), \ldots, B_{mn}(s) \rangle, \tag{5.19}$$

with

$$B_{kn}(s) = \sum_{j=0}^{n} \hat{v}_k \left(\frac{j}{n} \right) C_j^n s^j (1-s)^{n-j}, \tag{5.20a}$$

$$\hat{v}_k(s) = \frac{2v_k(s)}{1 + \sqrt{1 + 4\sum_{k=0}^{n} v_k^2(s)}}, \qquad k = 1, 2, \ldots, m, \tag{5.20b}$$

is N unboundedly compatible with $V(s)$.

(ii) *The vector approximant* $W(V, n, s)$ *converge to* $V(s)$ *uniformly on* $[0, 1]$ *as* $n \to \infty$, *in the parabolic metric.*

(iii) *On every closed subset* $I \subset [0, 1]$, *such that* $V(s)$ *is continuous we have*

$$\lim_{n\to\infty} \sup_{s\in I} ||V(s) - W(V, n, s)|| = 0. \tag{5.21}$$

Proof. We will prove only (i) of the proposition as the proof of the other parts is obvious. We will show that $\sum_{k=1}^{m} B_{kn}^2(s) = 1$ is impossible if $n > N$. To this end define two quantities J_1 and J_2, such that

$$\sum_{k=1}^{m} B_{kn}^2(s) = \sum_{k=1}^{m} \left[\sum_{j=0}^{n} \hat{v}_k\left(\frac{j}{n}\right) C_j^n s^j (1-s)^{n-j} \right]^2 = J_1 + J_2, \tag{5.22}$$

where

$$\begin{aligned} J_1 &= \sum_{k=1}^{m} \sum_{j=0}^{n} \hat{v}_k^2\left(\frac{j}{n}\right) [C_j^n s^j (1-s)^{n-j}]^2 \\ &= \sum_{j=0}^{n} [C_j^n s^j (1-s)^{n-j}]^2 \sum_{k=1}^{m} \hat{v}_k^2\left(\frac{j}{n}\right) \end{aligned} \tag{5.23}$$

and

$$J_2 = 2\sum_{k=1}^{m} \sum_{j_1 \neq j_2} \hat{v}_k\left(\frac{j_1}{n}\right) \hat{v}_k\left(\frac{j_2}{n}\right) C_{j_1}^n C_{j_2}^n s^{j_1} s^{j_2} (1-s)^{n-j_1} (1-s)^{n-j_2}. \tag{5.24}$$

The second sum in (5.24) is taken over all indices j_1 and j_2, such that $j_1, j_2 = 0, 1, \ldots, n$, and $j_1 \neq j_2$. It is readily observed that after the change of order of summation in J_2 we obtain

$$J_2 = 2\sum_{j_1 \neq j_2} C_{j_1}^n C_{j_2}^n s^{j_1} s^{j_2} (1-s)^{n-j_1} (1-s)^{n-j_2} \sum_{k=1}^{m} \hat{v}_k\left(\frac{j_1}{n}\right) \hat{v}_k\left(\frac{j_2}{n}\right). \tag{5.25}$$

By virtue of the Cauchy–Schwarz inequality we have

$$\sum_{k=1}^{m} \left| \hat{v}_k\left(\frac{j_1}{n}\right) \hat{v}_k\left(\frac{j_2}{n}\right) \right| \leqslant \left[\sum_{k=1}^{m} \hat{v}_k^2\left(\frac{j_1}{n}\right) \right]^{\frac{1}{2}} \left[\sum_{k=1}^{m} \hat{v}_k^2\left(\frac{j_2}{n}\right) \right]^{\frac{1}{2}}. \tag{5.26}$$

Consequently,

$$|J_2| \leqslant \sum_{j=0}^{n} [C_j^n s^j (1-s)^{n-j}]^2 \left[\sum_{k=1}^{m} \hat{v}_k^2\left(\frac{j_1}{n}\right) \right]^{\frac{1}{2}} \left[\sum_{k=1}^{m} \hat{v}_k^2\left(\frac{j_2}{n}\right) \right]^{\frac{1}{2}}. \tag{5.27}$$

We now make a few observations. We have $\sum_{k=1}^{m} \hat{v}_k^2(s) \leqslant 1$ and equality holds if $\sum_{k=1}^{m} v_k^2(s) = \infty$, for some value s in [0, 1]. Consequently, by (5.23) and (5.27) we have

$$\begin{aligned} \sum_{k=1}^{m} B_{kn}^2(s) = J_1 + J_2 \leqslant & \sum_{j=0}^{n} [C_j^n s^j (1-s)^{n-j}]^2 \\ & + 2\sum_{j_1 \neq j_2} C_{j_1}^n C_{j_2}^n s^{j_1} s^{j_2} (1-s)^{n-j_1} (1-s)^{n-j_2} \end{aligned}$$

$$\times\left[\sum_{k=1}^{m}\hat{v}_k^2\left(\frac{j_1}{n}\right)\right]^{\frac{1}{2}}\left[\sum_{k=1}^{m}\hat{v}_k^2\left(\frac{j_2}{n}\right)\right]^{\frac{1}{2}}$$

$$\equiv\left[\sum_{j=0}^{n}C_j^n s^j(1-s)^{n-j}\right]^2=1. \tag{5.28}$$

Moreover, the equality $\sum_{k=1}^{m} B_{kn}^2(s) \equiv 1 = [1-s+s]^{2n}$, holds if $\sum_{k=1}^{m}\hat{v}_k^2(\frac{j}{n}) = 1$, or if $\sum_{k=1}^{m} v_k^2(\frac{j}{n}) = \infty$, for all $j = 0, 1, 2, \ldots, n$. This is impossible if $n > N$ and the result follows. $\square$

Remark 11. The compactification presented here employs a bijection that is a C^∞ function. There are simpler compactifications like $\frac{z}{1+\|z\|}$, that could achieve similar results. The main difficulty in working with compactifications like $\frac{z}{1+\|z\|}$ is that they are not smooth enough. The mapping belongs to C^1 but not to C^k, $k > 1$, when $n = 1$, and is not smooth for $n > 1$. This could become a serious theoretical and practical handicap. Notice also that if z is a polynomial then $\frac{z}{1+\|z\|}$ could be either $\frac{z}{1+z}$ or $\frac{z}{1-z}$. Namely, two different polynomials may be involved.

Remark 12. Monotone approximation operators, that include the Bernstein operator as a particular case, could lead to other interesting unboundedly compatible approximations. Other positive kernels, that include the kernel in (5.17) as a particular case, could achieve similar goals.

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